HERC CyberSeminar
Interaction Terms in Nonlinear Models

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Introduction

- Health services researchers often use **interaction terms** in models with binary dependent variables
- Examples
  - Mortality depends on age, comorbidities (and interaction)
  - Readmission rate depends on nursing turnover rate, CQI program (and interaction)
  - Difference-in-differences models depend on Treatment-Control, Pre-post (and interaction)

Nonlinear Models

- Interaction terms are hard to interpret
- OLS intuition is misleading

1. Magnitude does not equal coefficient on interaction term (or even usual marginal effect)
2. Conditional on the independent variables (same as marginal effect of one variable)
3. Statistical significance is not *z*-statistic on interaction term
4. Sign may be different **(!)**

Outline

- OLS example with interaction term
- Logit example with marginal effect, 1 variable
- 2 logit examples with interaction terms
- Stata code
- Advanced stuff

Poll Question

- Which best describes your comfort with interaction terms and logistic regression?

1. I teach quantitative methods, very familiar
2. I write papers that use interaction terms
3. I read papers that use interaction terms
4. What are interaction terms?

Linear Models (OLS)

- Easy to compute marginal effects (for continuous variables) or incremental effects (for dummy variables)
- Coefficient on the interaction term gives the sign and magnitude of interaction effect
- Use *t*-test for statistical significance

OLS Example

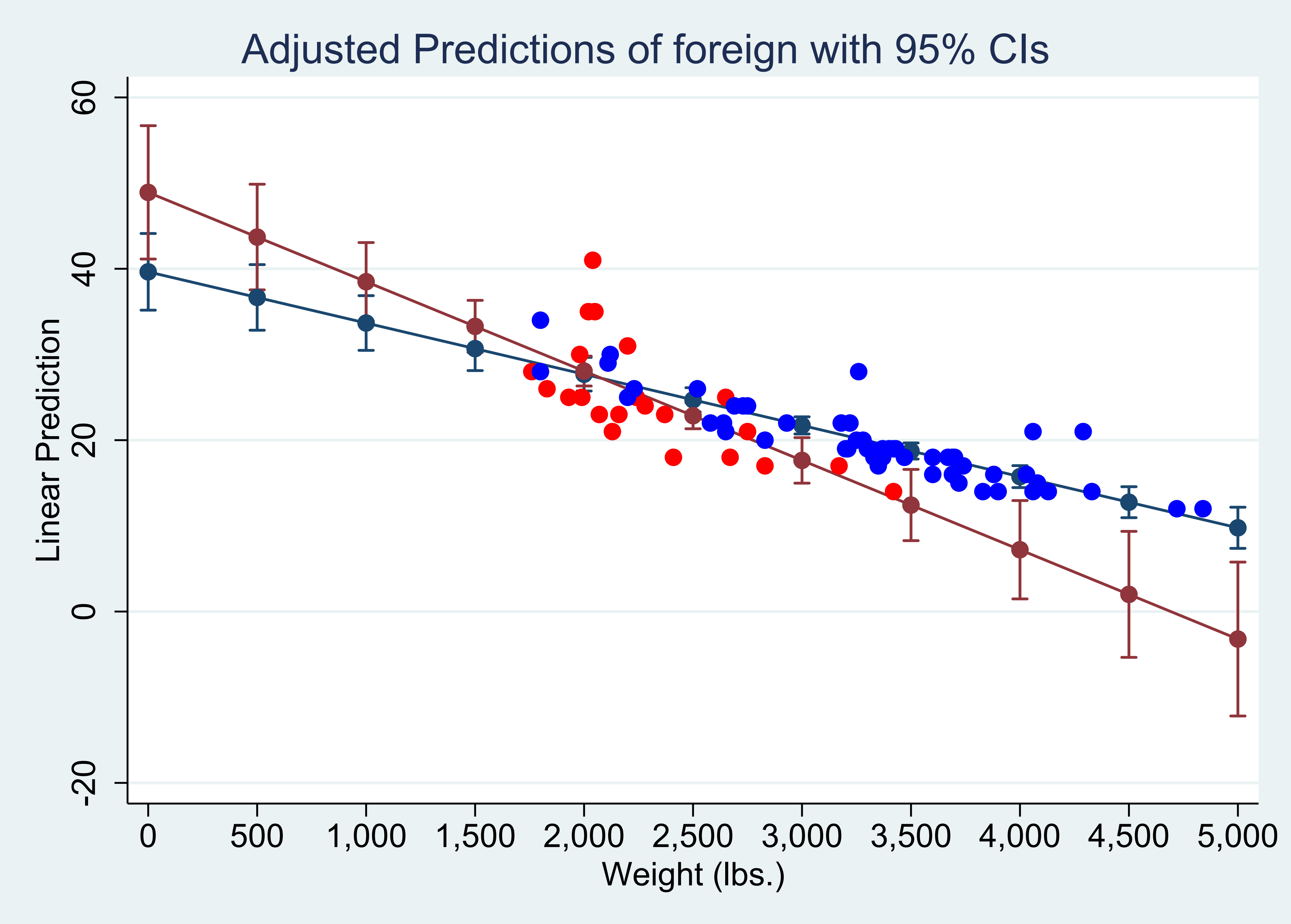
- Stata's automobile data set (`webuse auto`)
- N = 74, year is 1978
- Dependent variable is `mpg`
- Mean of mpg = 21.3

- mpg is function of weight (–), foreign (+)

Graphing Interaction Term

- Regress mpg on weight: 1 straight line
- Regress mpg on weight, foreign: 2 parallel lines
- Regress mpg on weight, foreign, and weight×foreign: 2 nonparallel lines

- `regress mpg c.weight##i.foreign`


Adjusted Predictions of foreign with 95% CIs
Linear Prediction
60
40
20
0
-20
0
500
1,000
1,500
2,000
2,500
3,000
3,500
4,000
4,500
5,000
Weight (lbs.)

Regression Output

```
• ---------------------------------------------------------------
•            mpg |      Coef.    Std. Err.        t
• ---------------+-----------------------------------------------
•         weight |  -.0059751    .0006622      -9.02
•                |
•        foreign |
•       Foreign  |   9.271333    4.500409       2.06
•                |
•       foreign#|
•       c.weight |
•       Foreign  |  -.0044509    .0017846      -2.49
•                |
•          _cons |   39.64696    2.243364      17.67
• ---------------------------------------------------------------
```

Interaction Term Interpretation

- What does –.00445 mean?
- *The marginal effect of weight is lower for foreign cars than for domestic cars by almost half an mpg per 100 lb. increase in weight*
- ME(weight|domestic) = –.00598
- ME(weight|foreign) = –.00598 + $\boxed{–.00445}$
- Coefficient tells us magnitude, sign
- *t*-statistic (–2.49) indicates significance at 5%

Math = Foreshadowing

- $mpg = \beta_0 + \beta_1 weight + \beta_2 foreign + \boxed{\beta_{12}} weight \times foreign + \varepsilon$
- Marginal effect = derivative = slope
- $ME(weight) = \beta_1 + \beta_{12} foreign$
- $ME(foreign) = \beta_2 + \beta_{12} weight$
- Interaction effect = **double derivative** = Δslope
- $\boxed{IE = \beta_{12}}$

Difference-in-Differences Models

- Common study design for new policy
- Pre-Post and Treatment-Control
- Two dummy variables and their interaction

- $Outcome = \beta_0 + \beta_1 Post + \beta_2 Tx + \boxed{\beta_{12}} Post \times Tx + \varepsilon$

OLS Interpretation of DD Models

	Pre	Post	**Difference**
Control	β_0	$\beta_0 + \beta_1$	β_1
Treatment	$\beta_0 + \beta_2$	$\beta_0 + \beta_1 + \beta_2 + \beta_{12}$	$\beta_1 + \beta_{12}$
Difference	β_2	$\beta_2 + \beta_{12}$	$\boxed{\beta_{12}}$

OLS Summary

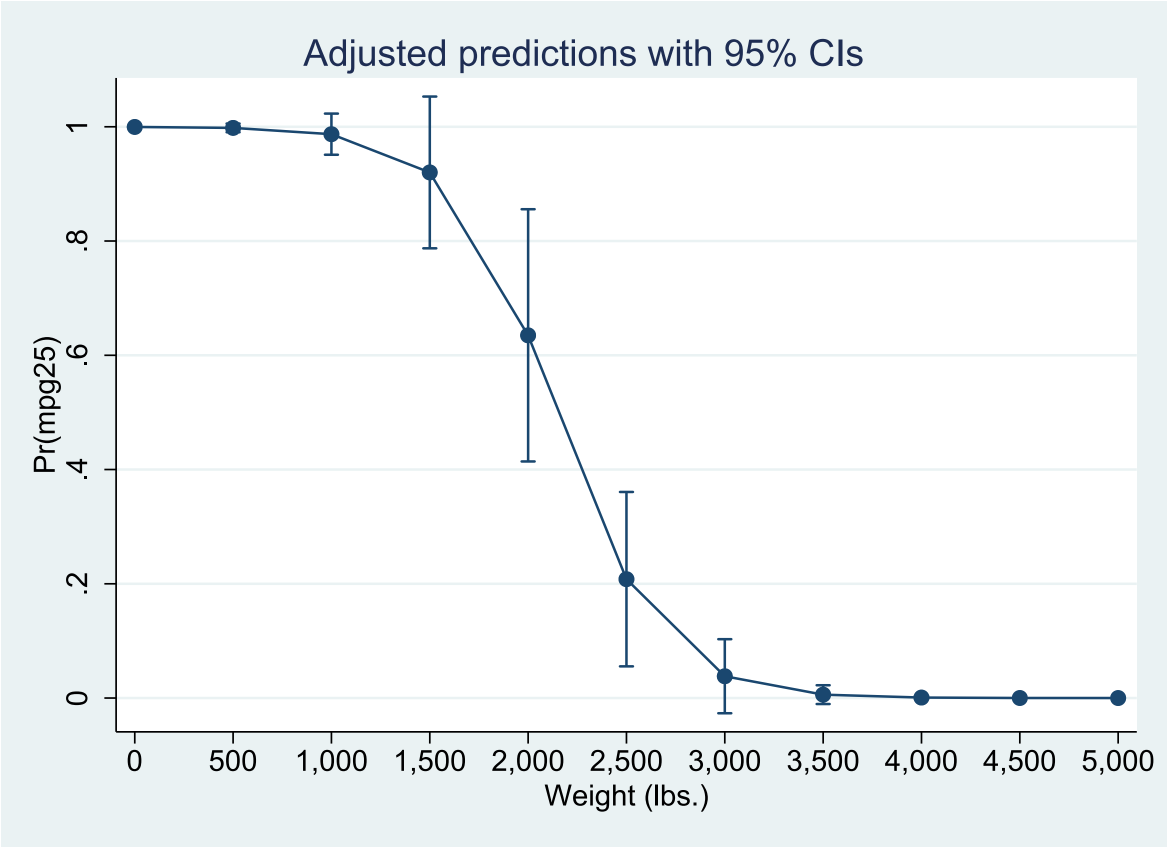
- Interaction effect is coefficient on interaction term
- Interaction effect is β_{12}
- Magnitude and sign are straightforward
- Significance is t-statistic on β_{12}

Marginal Effect of Single Variable

- More complicated in nonlinear models
- Not constant
- Vary with covariates
  - Summarize by taking ave. (“recycled predictions”)
- Smaller when the overall probability is small

Logit Example

- Let dependent variable indicate *if* mpg > 25
- Estimate logistic regression on just weight



Marginal Effect Formulas

- $ME = \beta_k \times pdf$ in general
- $ME = \beta_k \times F \times (1 - F)$ if logit
- $ME = \beta_k$□ if probit

- Fun fact: in logit $ME = \beta_k p(1 - p)$

Interaction Effects in Nonlinear Models

- General principles
  - Compute double difference or double derivative (or one of each)
  - Expect values to differ for each observation
  - Take average of interaction effects

General Formula

- ***Interaction effect*** is double difference or double derivative $(v = x\beta)$

$$\frac{\partial^2 E(y|x_1, x_2)}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_2}\left[\frac{dF}{dv}(\beta_1 + \beta_{12}x_2)\right]$$

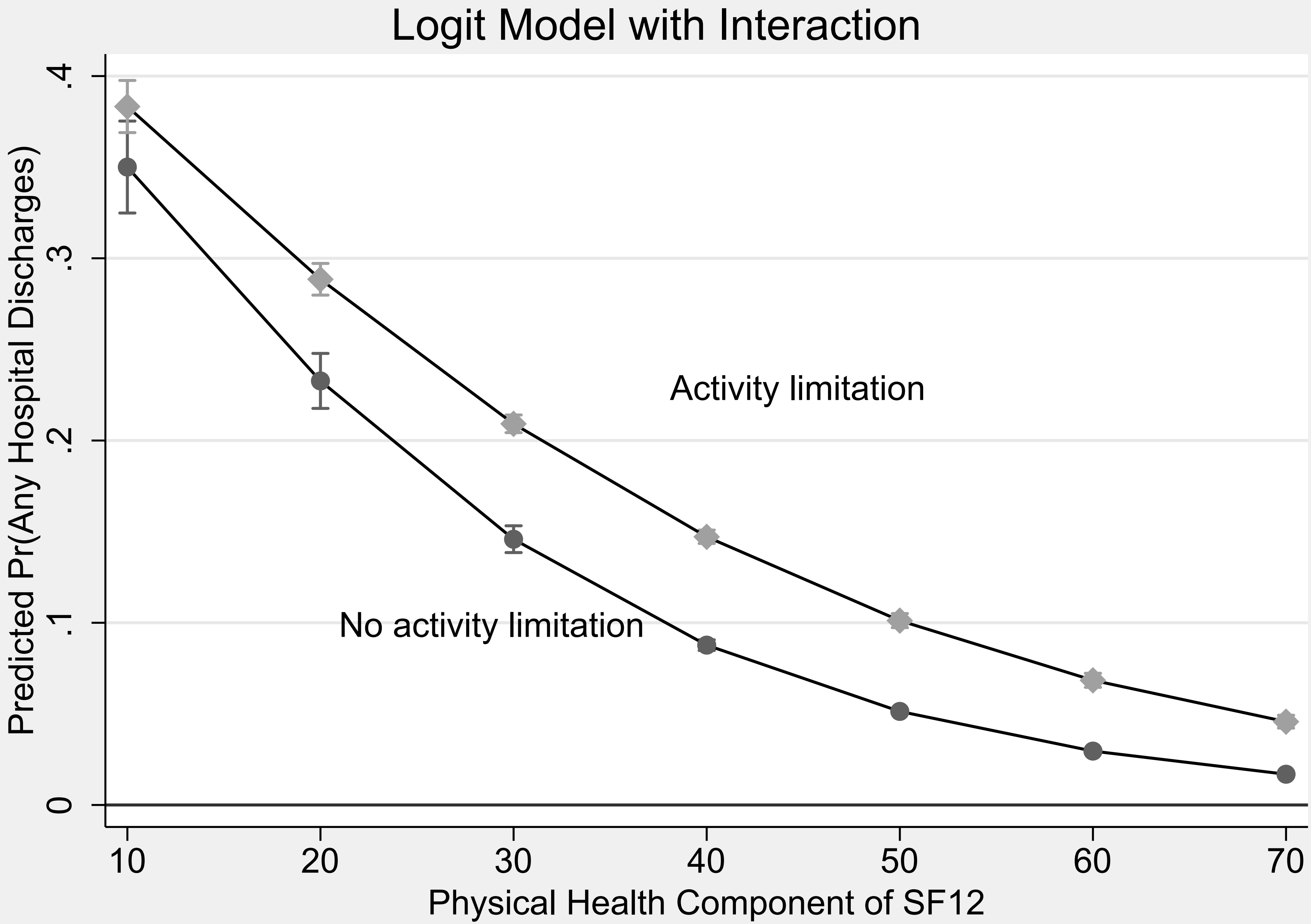
$$= \left[\frac{dF}{dv}\beta_{12}\right] + \left[\frac{d^2F}{dv^2}(\beta_1 + \beta_{12}x_2)(\beta_2 + \beta_{12}x_1)\right]$$

Interpretation of nonlinear DD

	Pre	Post	**Difference**
Control	$F(\beta_0)$	$F(\beta_0+\beta_1)$	$F(\beta_0+\beta_1)$ $-F(\beta_0)$
Treatment	$F(\beta_0+\beta_2)$	$F(\beta_0+\beta_1+\beta_2+\beta_{12})$	$F(\beta_0+\beta_1+\beta_2+\beta_{12})$ $-F(\beta_0+\beta_2)$
Difference	$F(\beta_0+\beta_2)$ $-F(\beta_0)$	$F(\beta_0+\beta_1+\beta_2+\beta_{12})$ $-F(\beta_0+\beta_1)$	$F(\beta_0+\beta_1+\beta_2+\beta_{12})$ $-F(\beta_0+\beta_2)$ $-F(\beta_0+\beta_1)$ $+F(\beta_0)$

Logit Example with Interaction (1)

- MEPS data from 2008–2014
- One observation per person, *N*=159,000
- Dependent variable: any hospital discharge?
  - Mean = 7.7%
- Function of
  - Any limitations (25% yes) (+)
  - Continuous health measure PCS (–)
  - Interaction



Logit Model with Interaction
Predicted Pr(Any Hospital Discharges)
0
.1
.2
.3
.4
Activity limitation
No activity limitation
10
20
30
40
50
60
70
Physical Health Component of SF12

Results

```
------------------------------------------------------------
             |               Robust
   any_disch | Coefficient  std. err.      z
-------------+----------------------------------------------
      anylim |  -.0041842   .0813945    -0.05
             |
         pcs | -.0574369     .001401   -41.00
             |
     anylim#|
         pcs |   .0147264   .0017849     8.25
             |
       _cons |  -.0443534   .0702603    -0.63
------------------------------------------------------------
```

Interaction Effect

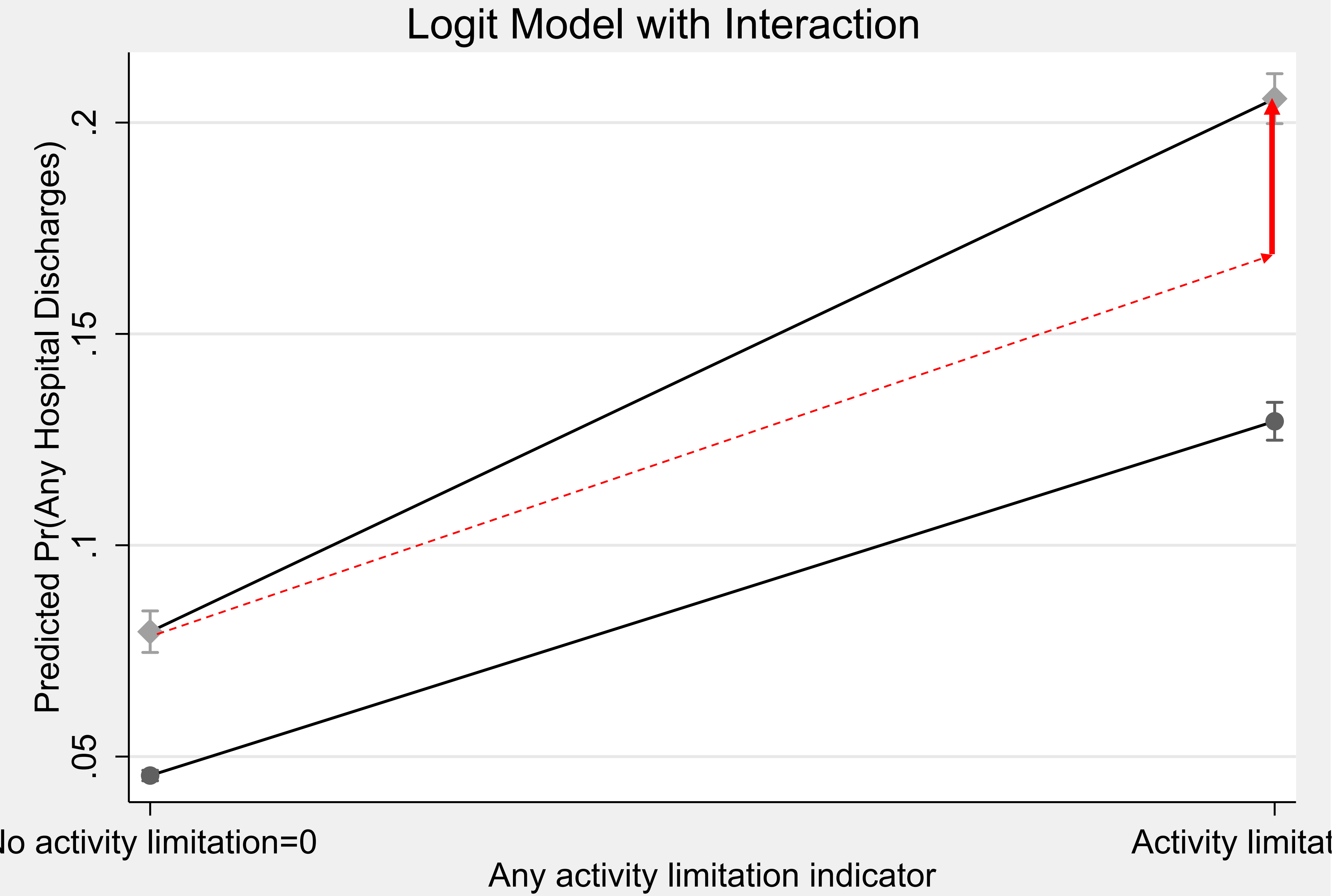
```
• margins, dydx(anylim) at(pcs=generate(pcs)) ///
•        at(pcs=generate(pcs + 1)) pwcompare(effect)
• ---------------------------------------------
•              |   Contrast Delta-method
•              |      dy/dx   std. err.      z
• -------------+--------------------------------
• 0.anylim     |  (base outcome)
• -------------+--------------------------------
• 1.anylim     |
•          _at |
•      2 vs 1  |  -.0008499   .0001251    -6.80
• ----------------------------------------------
```

Results

- *The marginal effect of an improvement in physical health is slightly lower for those with limitations than those without, when averaged over the sample*
- Most of sample has PCS between 40–60

Logit Example with Interaction (2)

- MEPS data from 2008–2014
- One observation per person, *N*=159,000
- Dependent variable: any hospital discharge?
  - Mean = 7.7%
- Function of 2 dichotomous variables
  - Any limitations (25% yes) (+)
  - Medicare coverage (19% yes) (+)
  - Interaction



Logit Model with Interaction
Predicted Pr(Any Hospital Discharges)
.05
.1
.15
.2
No activity limitation=0
Activity limitat
Any activity limitation indicator

Results

```
-------------------------------------------------------------
             |               Robust
   any_disch | Coefficient  std. err.       z
-------------+-----------------------------------------------
      anylim |   1.136398   .0250043     45.45
             |
    medicare |    .594836   .0372228     15.98
             |
      anylim#|
    medicare |  -.0392444   .0462375     -0.85
             |
       _cons |  -3.043308   .0146134   -208.25
-------------------------------------------------------------
```

Interaction Effect

```
• margins, dydx(anylim) at(medicare = (0 1))
•          pwcompare(effect)
• ---------------------------------------------
•                  |   Contrast Delta-method
•                  |      dy/dx   std. err.        z
• -------------+--------------------------------
• 0.anylim        |  (base outcome)
• -------------+--------------------------------
• 1.anylim        |
•           _at |
•      2 vs 1   |   .0422834   .0045824     9.23
• ---------------------------------------------
```

Meaning

- *The incremental effect of Medicare is 4 percentage points higher for those with any limitations than for those without*
- *The incremental effect of having any limitations is 4 percentage points higher for those on Medicare than for those not on Medicare*

Standard Errors

- Use Delta method for standard errors
- Provides no intuition, no point in deriving here
- See paper (Ai & Norton, 2003) for details
- Let Stata compute them for you

Stata Code

- `* Interaction effect for 1 binary & 1 continuous`
- `logit any_disch i.anylim##c.pcs, vce(robust)`
- `margins, dydx(anylim) at(pcs=generate(pcs)) at(pcs=generate(pcs + 1)) pwcompare(effect)`

- `* Interaction effect with 2 binary variables`
- `logit any_disch i.anylim##i.medicare, vce(robust)`
- `margins, dydx(anylim) at(medicare = (0 1)) pwcompare(effect)`

- `* Interaction effect with 2 continuous variables`
- `logit any_disch c.pcs##c.age, vce(robust)`
- `margins, dydx(pcs) at(age = generate(age)) at(age=generate(age + 1)) pwcompare(effect)`

Interpretation

- Greene (2010) argues that statistical testing should be for model building and specification
  - Then inform reader of predictions and marginal effects, use graphical analysis
- Puhani (2012) argues that if one cares about *treatment effect on the treated* (ATT), as opposed to average treatment effect (ATE), then only need interaction coefficient

Extensions

- Applies to all nonlinear models
  - Ordered and multinomial logit and probit
  - Count models
  - Follow same logic: take double derivatives of differences
- Triple interactions (including DDD models)
  - Follow same logic: take triple derivatives or differences

Linear Probability Model

- LPM is OLS with dummy dependent variable
- Interaction effects are as simple as in OLS
- Problems with LPM
  - Predictions may be outside [0,1] interval
  - Assumes constant marginal effect
- May prefer LPM if care about overall average
- May prefer LPM if model has fixed effects
- Suggestion: estimate both and compare

Conclusions

- Interaction effects are more complicated in nonlinear models than in OLS
- Only looking at coefficient on the interaction term is wrong:
  - Wrong magnitude
  - Wrong statistical significance
  - Wrong sign (perhaps!)
- Our papers have formulas and examples

References (1)

- Ai & Norton. 2003. Interaction terms in logit and probit models. ***Economics Letters*** 80(1):123–129.
- Norton, Wang, & Ai. 2004. Computing interaction effects in logit and probit models. ***The Stata Journal*** 4(2):154–167.
- Karaca-Mandic, Norton, & Dowd. 2012. Interaction terms in nonlinear models. ***Health Services Research*** 47(1):255–274.

References (2)

- Greene. 2010. Testing hypotheses about interaction terms in nonlinear models. ***Economics Letters*** 107(2):291-296.
- Puhani. 2012. The treatment effect, the cross difference, and the interaction term in nonlinear "difference-in-differences" models. ***Economics Letters*** 115(1):85-87.

Thank You!

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